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Quantum-Symmetric and Quantum-Antisymmetric Matrices Corresponding to a Quantum Group and Quadratic Homogeneous Expressions on Quantum Hyperplane

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Quantum-symmetric matrices and quantum-antisymmetric matrices corresponding to a given quantum group are discussed, some concrete examples are given, and some relevant invariances are proven.

In physics and mathematics symmetric and antisymmetric matrices play important roles, and in linear algebra symmetric matrices, antisymmetric matrices, coordinate transformations, and quadratic homogeneous forms are related to each other. In quantum groups and noncommutative geometry (Manin, 1988; Faddeev *et al.*, 1987), when the quantum deformation parameters approach 1, a quantum group G_q can in fact be regarded as an ordinary linear group G. Therefore we expect that in quantum groups and noncommutative geometry, there are also quantum-symmetric matrices and quantum-antisymmetric matrices. This paper gives some fundamental results for this problem. We find that the case is different from ordinary linear algebra. There is not a unitary form for the quantum-symmetric matrix or quantum-antisymmetric matrix in quantum groups and noncommutative geometry. In fact, in ordinary linear algebra whenever a matrix $M = (M^i_j)$ is in one of the matrix groups, if $M^i_j = M^j_i$, then M is called symmetric, or if $M^i_j = -M^j_i$, then M is called antisymmetric, and the symmetry or antisymmetry is invariant under an involution transformation of matrices, etc. However, for different quantum groups G_q, the G_q-symmetric or G_q-antisymmetric matrices have different forms, which depend on the choice of the expressions for the commu-

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tation relation in these quantum groups. The quantum symmetry is also invariant under an involution transformation of quantum matrices. In particular, it is interesting that in the ordinary sense these G_q-symmetric or G_q-antisymmetric matrices, on the contrary, are not symmetric or antisymmetric, respectively. Only when the quantum deformation parameters approach 1 do they change into ordinary symmetric or antisymmetric matrices, respectively.

In this paper we use the following known results. Suppose that G_q is a quantum group whose elements are $n \times n$ quantum matrices $M = (M^i_j)$; then there is a Yang–Baxter matrix $\check{R} = (\check{R}^{ij}_{kl})$ $(i, j, k, l = 1, 2, \ldots, n)$ corresponding to G_q, satisfying the Yang–Baxter equation

$$\check{R}_{12}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}\check{R}_{23} \tag{1}$$

The action space of G_q is a quantum hyperplane P, which, in fact, is a noncommutative and associative $\mathbb{C}$- (or any field of characteristic zero) algebra with the unit element. Let x^i $(i = 1, 2, \ldots, n)$ denote the coordinates (the generators). We require that in P there is the differential calculus: the partial derivative operator $\partial_i = \partial/\partial x^i$, $\partial_i(x^j) = \delta^j_i$, and the differential $dx^i\partial_i$. According to Song and Liao (1992, 1993), the commutation relations among x^i, ∂_j, and dx^i are

$$x^i x^j = B^{ij}_{kl} x^k x^l \tag{2a}$$

$$x^i dx^j = C^{ij}_{kl} dx^k x^l \tag{2b}$$

$$\partial_k x^i = \delta^i_k + C^{ij}_{kl} x^k \partial_j \tag{2c}$$

$$dx^i dx^j = -(C^{-1})^{ij}_{kl} dx^k dx^l \tag{2d}$$

$$\partial_k dx^i = (C^{-1})^{ij}_{kl} dx^l \partial_j \tag{2e}$$

$$\partial_i \partial_j = B^{lk}_{ji} \partial_k \partial_l \tag{2f}$$

where B and C are numerical matrices. The consistency of equations (2a)–(2f) is guaranteed by the following equations:

$$\begin{aligned} (E_{12} - B_{12})(E_{12} + C_{12}) &= 0 \\ (E_{12} - B_{12})C_{23}C_{12} &= C_{23}C_{12}(E_{23} - B_{23}) \\ C_{12}C_{23}C_{12} &= C_{23}C_{12}C_{23} \end{aligned} \tag{3}$$

where E is the unit matrix. The concrete values of B and C can be chosen according to the corresponding $\check{R} = (\check{R}^{ij}_{kl})$. In this paper, we assume that B and C have been determined for P. A quantum matrix $M \in G_q$ means that M satisfies the Yang–Baxter relation

$$M_1 M_2 \check{R}_{12} = \check{R}_{12} M_1 M_2 \tag{4}$$

The commutation relations (2a–2f) are covariant under the following transformations:

$$x^i \to \tilde{x}^i = M^i_j x^j, \qquad dx^i \to d\tilde{x}^i = M^i_j\, dx^j$$
$$\partial_i \to \tilde{\partial}_i = [(M^t)^{-1}]^k_i \partial_k \tag{5}$$

where t denotes the transposition of a matrix. Equations (5) give

$$(E_{12} - B_{12})M_1 M_2 = 0, \qquad (E_{12} + C^{-1}_{12})M_1 M_2 = 0 \tag{6}$$

Therefore the quantum group G_q can be regarded a "coordinate transformation group" of the quantum hyperplane P.

Now we consider what is a quantum-symmetric matrix. If a_i ($i = 1, 2, \ldots, n$) are n algebraic elements, and $a_i a_j = a_j a_i$ (e.g., if $a_i \in \mathbb{C}$), then the matrix $N = (N_{ij})$, $N_{ij} = a_i a_j$, obviously is a symmetric matrix in the ordinary sense. However, if a_i does not commute with a_j for $i \neq j$, and there is a consistent commutation relation $a_i a_j = Z^{kl}_{ij} a_k a_l$, where Z is a numerical matrix containing some deformation parameters, then N is not a symmetric matrix in the ordinary sense. However, the commutation relation $N_{ab} = Z^{rs}_{ab} N_{rs}$ reflects some "quantum symmetry" of N; in other words, N is a quantum deformation of an ordinary symmetric matrix. For a quantum group G_q, the quantum G_q symmetry must be invariant under the linear transformations introduced by transformations as in equations (5); therefore the choice of the numerical matrix Z is restricted. In addition, for the quantum group G_q such matrices, in fact, cannot consist of x^i, ∂_j, and dx^k, and we must reconsider other noncommutative algebras.

Definition 1. If an $n \times n$ matrix $\mathcal{L} = (\mathcal{L}_{ab})$ ($a, b = 1, 2, \ldots, n$) satisfies the relation

$$\mathcal{L}_{ab} = B^{ij}_{ab} \mathcal{L}_{ij} \tag{7}$$

where B is defined as in equations (2a), (2f), and (3), then $\mathcal{L}$ is called a (quantum) G_q-symmetric matrix.

Remark. The $\mathcal{L}_{ab}$ can be both numbers and abstract algebraic elements, in the latter case there may be other commutation relations, and in this way some quantum groups can be nonlinearly realized (Zhong, 1994, 1996). In addition, such a G_q-symmetric matrix obviously is not a symmetric matrix in the ordinary sense unless all quantum deformation parameters approach 1 (then $B^{ij}_{ab} = \delta^i_a \delta^j_b$).

Proposition 1. The G_q symmetry is invariant under a G_q-involution transformation, i.e., if $\mathcal{L}$ is G_q-symmetric and $M \in G_q$, then $\tilde{\mathcal{L}} = M^t \mathcal{L} M$ is also G_q-symmetric.

Proof. According to equations (6) and (7),

$$B^{ij}_{ab}\tilde{\mathcal{L}}_{ij} = B^{ij}_{ab}M^r_iM^s_j\mathcal{L}_{rs} = M^r_aM^s_b\mathcal{L}_{rs} = \tilde{\mathcal{L}}_{ab} \tag{8}$$

so $\tilde{\mathcal{L}}$ is G_q-symmetric. QED

In order to write the concrete form of a G_q-symmetric matrix $\mathcal{L}$, the relation between $\mathcal{L}_{ab}$ for $a < b$ and $\mathcal{L}_{ji}$ for $j \geq i$ is required. Let us write equation (7) as

$$\mathcal{L}_{ab} = B^{i<j}_{ab}\mathcal{L}_{i<j} + B^{i\geq j}_{ab}\mathcal{L}_{i\geq j} \tag{9}$$

where here and in the following we use simplified symbols, e.g., $B^{i<j}_{ab}$ denotes those B^{ij}_{ab} when $i < j$, and $B^{i<j}_{ab}\mathcal{L}_{i<j} \equiv \Sigma_{i<j} B^{ij}_{ab}\mathcal{L}_{ij}$, etc. Thus we can obtain

$$(\mathcal{L}_{b>a})^t L^{b>a}_{j>i} = (\mathcal{L}_{r\leq s})^t B^{r\leq s}_{j>i}$$

$$(\mathcal{L}_{a<b})^t L^{a<b}_{i<j} = (\mathcal{L}_{s\geq r})^t B^{s\geq r}_{i<j} \tag{10}$$

where $(\mathcal{L}_{a<b})^t$ is the $\frac{1}{2}n(n - 1)$ line matrix $(\mathcal{L}_{12}, \ldots, \mathcal{L}_{1n}, \mathcal{L}_{23}, \ldots, \mathcal{L}_{n-1,n})$, etc., and

$$L^{a<b}_{i<j} = \delta^a_i\delta^b_j - B^{a<b}_{i<j}, \qquad L^{b>a}_{j>i} = \delta^b_j\delta^a_i - B^{b>a}_{j>i} \tag{11}$$

If we take equations (10) as linear equations about the unknown $\mathcal{L}_{a<b}$ or $\mathcal{L}_{b>a}$ and solve them, then we obtain the required expressions as follows:

$$\mathcal{L}_{a<b} = \mathcal{L}_{s\geq r}\Lambda^{s\geq r}_{a<b}, \qquad \Lambda^{s\geq r}_{a<b} = B^{s\geq r}_{i<j}(L^{-1})^{i<j}_{a<b}$$

$$\mathcal{L}_{b>a} = \mathcal{L}_{r\leq s}\Gamma^{r\leq s}_{b>a}, \qquad \Gamma^{r\leq s}_{b>a} = B^{r\leq s}_{j>i}(L^{-1})^{j>i}_{b>a} \tag{12}$$

where we have used $\det(L) \neq 0$; for a given quantum group G_q, this condition generally can be satisfied by some suitable choices of B. Therefore the general form of a G_q-symmetric matrix is

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \cdots & \mathcal{L}_{1n} \\ \Gamma^{r\leq s}_{21}\mathcal{L}_{r\leq s} & \mathcal{L}_{22} & \cdots & \mathcal{L}_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \Gamma^{r\leq s}_{n1}\mathcal{L}_{r\leq s} & \Gamma^{r\leq s}_{n2}\mathcal{L}_{r\leq s} & \cdots & \mathcal{L}_{nn} \end{pmatrix} = \begin{pmatrix} \mathcal{L}_{11} & \Lambda^{s\geq r}_{12}\mathcal{L}_{s\geq r} & \cdots & \Lambda^{s\geq r}_{1n}\mathcal{L}_{s\geq r} \\ \mathcal{L}_{21} & \mathcal{L}_{32} & \cdots & \Lambda^{s\geq r}_{2n}\mathcal{L}_{s\geq r} \\ \cdots & \cdots & \cdots & \cdots \\ \mathcal{L}_{n1} & \mathcal{L}_{n2} & \cdots & \mathcal{L}_{nn} \end{pmatrix} \tag{13}$$

Example 1. The general quantum linear group is $GL_{X;q_{ij}}(N)$, where $q_{ij} \in \mathbb{C}$ $(i < j)$ are the quantum deformation parameters, and $X \in \mathbb{C}$, $p_{ij}q_{ij} =$

X. When all $p_{ij} = q_{ij} = 1$, then $GL_{X;q_{ij}}(N)$ changes into the ordinary $GL_q(N)$. Notice that $\check{R}^{ij}_{kl} = R^{ji}_{kl}$, then according to Schirrmacher (1991),

$$B^{ij}_{Kl} = (\check{R}_{X;q_{ij}})^{ij}_{kl} = \delta^j_k\delta^i_l\left(\delta^{ij} + \theta^{ij}\frac{1}{q_{ji}} + \theta^{ji}\frac{1}{p_{ij}}\right) + \delta^j_l\delta^i_k\theta^{ji}\left(1 - \frac{1}{X}\right) \tag{14}$$

where θ^{ij} equals 1 for $i > j$ and 0 otherwise. From (11) we obtain

$$\begin{aligned} L^{b>a}_{j>i} &= \delta^b_j\delta^a_i, & B^{r\leq s}_{k>l} &= \delta^s_k\delta^r_l\frac{1}{p_{rs}} \\ \Gamma^{r\leq s}_{k>l} &= \delta^s_k\delta^r_l\frac{1}{p_{rs}}, & \Lambda^{s\geq r}_{l<k} &= \delta^r_l\delta^s_k p_{rs} \end{aligned} \tag{15}$$

Therefore the general form of a $GL_{X;q_{ij}}(N)$-symmetric matrix is

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \cdots & \mathcal{L}_{1n} \\ (1/p_{12})\mathcal{L}_{12} & \mathcal{L}_{23} & \cdots & \mathcal{L}_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ (1/p_{1n})\mathcal{L}_{1n} & (1/p_{2n})\mathcal{L}_{2n} & \cdots & \mathcal{L}_{nn} \end{pmatrix} = \begin{pmatrix} \mathcal{L}_{11} & p_{12}\mathcal{L}_{21} & \cdots & p_{1n}\mathcal{L}_{n1} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \cdots & p_{2n}\mathcal{L}_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ \mathcal{L}_{n1} & \mathcal{L}_{n2} & \cdots & \mathcal{L}_{nn} \end{pmatrix} \tag{16}$$

To sum up, the $GL_{X;q_{ij}}(N)$-symmetry relation is $\mathcal{L}_{ba} = (1/p_{ab})\mathcal{L}_{ab}$ for $a < b$.

Example 2. The general expressions with regard to $SO_q(N)$ are more complicated; here we only do the calculation for $SO_q(3)$. Now we take (Song and Liao, 1992, 1993)

$$B = \frac{1}{q}\check{R} = \begin{array}{c|ccccccccc} & 11 & 12 & 13 & 21 & 22 & 23 & 31 & 32 & 33 \\ \hline 11 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 12 & 0 & 1-\frac{1}{q^2} & \frac{1}{q} & 0 & 0 & 0 & 0 & 0 & 0 \\ 13 & 0 & 0 & \left(1-\frac{1}{q^2}\right)\left(1-\frac{1}{q}\right) & 0 & -\left(1-\frac{1}{q^2}\right)\frac{1}{\sqrt{q}} & 0 & \frac{1}{q^2} & 0 & 0 \\ 21 & 0 & \frac{1}{q} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 22 & 0 & 0 & -\left(1-\frac{1}{q^2}\right)\frac{1}{\sqrt{q}} & 0 & \frac{1}{q} & 0 & 0 & 0 & 0 \\ 23 & 0 & 0 & 0 & 0 & 0 & 1-\frac{1}{q^2} & 0 & \frac{1}{q} & 0 \\ 31 & 0 & 0 & \frac{1}{q^2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 32 & 0 & 0 & 0 & 0 & 0 & \frac{1}{q} & 0 & 0 & 0 \\ 33 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \tag{17}$$

From (17) we obtain

$$B^{\leq}_{>} = \Gamma^{\leq}_{>} = \begin{array}{c} \\ 11 \\ 12 \\ 13 \\ 22 \\ 23 \\ 33 \end{array} \begin{array}{c} \begin{array}{ccc} 21 & 31 & 32 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{q} & 0 & 0 \\ 0 & \frac{1}{q^2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{q} \\ 0 & 0 & 0 \end{pmatrix} \end{array}, \quad B^{\geq}_{<} = \begin{array}{c} \\ 11 \\ 21 \\ 22 \\ 31 \\ 32 \\ 33 \end{array} \begin{array}{c} \begin{array}{ccc} 12 & 13 & 23 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{q} & 0 & 0 \\ 0 & -\left(1 - \frac{1}{q^2}\right)\frac{1}{\sqrt{q}} & 0 \\ 0 & \frac{1}{q^2} & 0 \\ 0 & 0 & \frac{1}{q} \\ 0 & 0 & 0 \end{pmatrix} \end{array}, \quad \Lambda^{\geq}_{<} = \begin{array}{c} \\ 11 \\ 21 \\ 22 \\ 31 \\ 32 \\ 33 \end{array} \begin{array}{c} \begin{array}{ccc} 12 & 13 & 23 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{(1-q^2)\sqrt{q}}{q^2+q-1} & 0 \\ 0 & \frac{q}{q^2+q-1} & 0 \\ 0 & 0 & q \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$L^{\geq}_{>} = \begin{array}{c} \\ 21 \\ 31 \\ 32 \end{array} \begin{array}{c} \begin{array}{ccc} 21 & 31 & 32 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}, \quad L^{\leq}_{<} = \begin{array}{c} \\ 12 \\ 13 \\ 23 \end{array} \begin{array}{c} \begin{array}{ccc} 12 & 13 & 23 \end{array} \\ \begin{pmatrix} \frac{1}{q^2} & 0 & 0 \\ 0 & \frac{1}{q} + \frac{1}{q^2} - \frac{1}{q^3} & 0 \\ 0 & 0 & \frac{1}{q^2} \end{pmatrix} \end{array} \tag{18}$$

Therefore the general form of a $SO_q(3)$-symmetric matrix is

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \frac{1}{q}\mathcal{L}_{12} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \frac{1}{q^2}\mathcal{L}_{13} & \frac{1}{q}\mathcal{L}_{23} & \mathcal{L}_{33} \end{pmatrix} = \begin{pmatrix} \mathcal{L}_{11} & q\mathcal{L}_{21} & \frac{(1-q^2)\sqrt{q}}{q^2+q-1}\mathcal{L}_{32} + \frac{q}{q^2+q-1}\mathcal{L}_{31} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & q\mathcal{L}_{32} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{pmatrix} \tag{19}$$

To sum up, the $SO_q(3)$-symmetry relations are

$$\mathcal{L}_{21} = \frac{1}{q}\mathcal{L}_{12}, \qquad \mathcal{L}_{32} = \frac{1}{q}\mathcal{L}_{23}, \qquad \mathcal{L}_{31} = \frac{1}{q^2}\mathcal{L}_{13}$$
$$\mathcal{L}_{13} - \mathcal{L}_{31} = \left(\sqrt{q} - \frac{1}{\sqrt{q}}\right)\mathcal{L}_{22} \tag{20}$$

In addition, it has been proved (Song and Liao, 1992, 1993) that in the q-Euclidean space E_q^3 one has the metric g_{ij} and

$$g_{ij}\check{R}^{ij}_{kl} = \lambda g_{kl} \tag{21}$$

where λ is an eigenvalue of $\check{R}$. Therefore the metric g_{ij}, in fact, is a $SO_q(3)$-symmetric numerical matrix.

To return to the general case, if $\mathcal{L} = (\mathcal{L}_{rs})$ is a G_q-symmetric matrix whose entries commute with the coordinates x^i of P, then $F = (x)^t\mathcal{L}(x) = \mathcal{L}_{rs}x^r x^s$ is a quadratic homogeneous expression on P. We call F a "standard quadratic homogeneous expression" on P.

Proposition 2. Under the coordinate transformation of P as in (5) a standard quadratic homogeneous expression on P is changed into a standard quadratic homogeneous expression on P.

Proof. Under the transformation as in (5)

$$F \to \tilde{F} = (\tilde{x})^t\mathcal{L}(\tilde{x}) = [M(x)]^t\mathcal{L}M(x) = (x)^t[M^t\mathcal{L}M](x) = (x)^t\tilde{\mathcal{L}}(x) \quad (22)$$

By Proposition 1, $\tilde{\mathcal{L}}$ still is G_q-symmetric, so $\tilde{F}$ is a standard quadratic homogeneous expression on P. QED

Proposition 3. Let the commutation relation (2a) be regarded as a substitution

$$x^i x^j \to B^{ij}_{kl} x^k x^l \quad (23)$$

Then under this substitution a standard quadratic homogeneous expression is invariant.

Proof. In fact, by (7)

$$F = \mathcal{L}_{rs}x^r x^s \to F' = \mathcal{L}_{rs}B^{rs}_{ij}x^i x^j = \mathcal{L}_{ij}x^i x^j = F \quad \text{QED} \quad (24)$$

In the following we discuss quantum-antisymmetric matrices.

Definition 2. If an $n \times n$ matrix $\mathcal{A} = (\mathcal{A}_{ab})$ $(a, b = 1, 2, \ldots, n)$ satisfies

$$\mathcal{A}_{ab} = -[C^{-1}]^{ij}_{ab}\mathcal{A}_{ij} \quad (25)$$

where C is defined as in (2c) and (3), then $\mathcal{A}$ is called a (quantum) G_q-antisymmetric matrix.

Remark. The $\mathcal{A}_{ab}$ can be both numbers and abstract algebraic elements. In the latter case, by the use of polynomials on P some quantum groups can be nonlinearly realized. In addition, a G_q-antisymmetric matrix is always not an antisymmetric matrix in the ordinary sense unless all quantum deformation parameters approach 1 (then $[C^{-1}]^{ij}_{ab} = \delta^i_a\delta^j_b$).

Proposition 4. The G_q-antisymmetry is invariant under a G_q-involution transformation, i.e., if $\mathcal{A}$ is G_q-antisymmetric and $M \in G_q$, then $\tilde{\mathcal{A}} = M^t\mathcal{A}M$ is also G_q-antisymmetric.

Proof. According to equations (6) and (25),

$$-[C^{-1}]^{ij}_{ab}\tilde{\mathcal{A}}_{ij} = -[C^{-1}]^{ij}_{ab}M^r_i M^s_j \mathcal{A}_{rs} = M^i_a M^j_b \mathcal{A}_{ij} = \tilde{\mathcal{A}}_{ab} \quad (26)$$

so $\tilde{\mathcal{A}}$ is G_q-antisymmetric. QED

Equation (25) can be written as

$$(\mathscr{A}_{b>a})^t K_{j>i}^{b>a} = (\mathscr{A}_{r\le s})^t [-C^{-1}]_{j>i}^{r\le s} \qquad (27)$$
$$K_{j>i}^{b>a} = \delta_j^b \delta_i^a + [-C^{-1}]_{j>i}^{b>a}$$

If equation (27) is regarded as a linear equation about $\mathscr{A}_{b>a}$, then we can obtain

$$\mathscr{A}_{b>a} = -\mathscr{A}_{r\le s} \Psi_{b>a}^{r\le s} \qquad (28)$$
$$\Psi_{b>a}^{r\le s} = [C^{-1}]_{j>i}^{r\le s} [K^{-1}]_{b>a}^{j>i}$$

Similarly, we have

$$\mathscr{A}_{a<b} = -\mathscr{A}_{s\ge r} \Phi_{a<b}^{s\ge r} \qquad (29)$$
$$\Phi_{a<b}^{s\ge r} = [C^{-1}]_{i<j}^{s\ge r} [K^{-1}]_{a<b}^{i<j}$$

Therefore, the general form of a G_q-antisymmetric matrix is

$$\mathscr{A} = \begin{pmatrix} \mathscr{A}_{11} & \mathscr{A}_{12} & \cdots & \mathscr{A}_{1n} \\ -\Psi_{21}^{r\le s}\mathscr{A}_{r\le s} & \mathscr{A}_{22} & \cdots & \mathscr{A}_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ -\Psi_{n1}^{r\le s}\mathscr{A}_{r\le s} & -\Psi_{n2}^{r\le s}\mathscr{A}_{r\le s} & \cdots & \mathscr{A}_{nn} \end{pmatrix} = \begin{pmatrix} \mathscr{A}_{11} & -\Phi_{12}^{s\ge r}\mathscr{A}_{s\ge r} & \cdots & -\Phi_{1n}^{s\ge r}\mathscr{A}_{s\ge r} \\ \mathscr{A}_{21} & \mathscr{A}_{22} & \cdots & -\Phi_{2n}^{s\ge r}\mathscr{A}_{s\ge r} \\ \cdots & \cdots & \cdots & \cdots \\ \mathscr{A}_{n1} & \mathscr{A}_{n2} & \cdots & \mathscr{A}_{nn} \end{pmatrix} \qquad (30)$$

Remark. As for the diagonal entries $\mathscr{A}_{aa}$ ($a = 1, 2, \ldots, n$), it is easily seen from $\mathscr{A}_{aa} = -[C^{-1}]_{aa}^{ij}\mathscr{A}_{ij}$ that many of the $\mathscr{A}_{11}, \mathscr{A}_{22}, \ldots, \mathscr{A}_{nn}$ in fact, are zeros for a concrete C.

Example 3. For $G_q = GL_{X;q_{ij}}(N)$ (Schirrmacher, 1991; Zhong, 1996), and $\check{R}$ as in (14), we can take $C^{-1} = X\check{R}$. Therefore the general form of a $GL_{X;q_{ij}}(N)$-antisymmetric matrix is

$$\mathscr{A} = \begin{pmatrix} 0 & \mathscr{A}_{12} & \cdots & \mathscr{A}_{1n} \\ -q_{12}\mathscr{A}_{12} & 0 & \cdots & \mathscr{A}_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ -q_{1n}\mathscr{A}_{1n} & -q_{2n}\mathscr{A}_{2n} & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 0 & -(1/q_{12})\mathscr{A}_{21} & \cdots & -(1/q_{1n})\mathscr{A}_{n1} \\ \mathscr{A}_{21} & 0 & \cdots & -(1/q_{2n})\mathscr{A}_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ \mathscr{A}_{n1} & \mathscr{A}_{n2} & \cdots & 0 \end{pmatrix} \qquad (31)$$

Example 4. For $G_q = SO_q(3)$, $\check{R}$ is as in (17). We can take $C^{-1} = q\check{R}$; then the general form of a $SO_q(3)$-antisymmetric matrix is

$$\mathcal{A} = \begin{pmatrix} 0 & \mathcal{A}_{12} & \mathcal{A}_{13} \\ -q\mathcal{A}_{12} & \mathcal{A}_{22} & \mathcal{A}_{23} \\ -\mathcal{A}_{13} & -q\mathcal{A}_{23} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\frac{1}{q}\mathcal{A}_{21} & \frac{1-q^2}{q\sqrt{q}(q^2+q-1)}\mathcal{A}_{32} - \frac{1}{q(q^2+q-1)}\mathcal{A}_{31} \\ \mathcal{A}_{21} & \mathcal{A}_{22} & -\frac{1}{q}\mathcal{A}_{32} \\ \mathcal{A}_{31} & \mathcal{A}_{32} & 0 \end{pmatrix} \tag{32}$$

If $\mathcal{A}$ is a general G_q-antisymmetric matrix, then we call $H = \mathcal{A}_{ab}dx^a dx^b$ a "standard quadratic form" on the quantum hyperplane P.

Proposition 5. Under the coordinate transformation of P as in (5), a standard quadratic form on P is changed into a standard quadratic form on P.

Proposition 6. Let the commutation relation (2d) be regarded as a substitution

$$dx^i dx^j \to -[C^{-1}]^{ij}_{kl}\, dx^k dx^l \tag{33}$$

Then under this substitution a standard quadratic form is invariant.

The proofs of Propositions 5 and 6 are similar to those of Propositions 2 and 3, respectively.

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